AN ANALYSIS OF ORDERED OBSERVATIONS IN BLOCK DESIGNS

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SUMMARY

A method of analysis of the the data which does not satisfy the requirements of analysis of variance technique, has been developed. A mathematical model has been formulated and test procedures are discussed. In the null hypothesis we assume that the treatment ratings are equal whereas the alternative hypothesis does not make any assumption of equality of treatment ratings. The probability of treatment preferences $P(T_1 > T_2 > \cdots > T_t)$ involves $\binom{t}{2}$ paired comparisons. By following the same approach we may develope the method of analysis in the similar line for incomplete block designs.

1. INTRODUCTION

The techniques of rank analysis are quite useful and these have been used for qualitative characters. Hotelling and Richards Pabst (1936) and Friedman (1937) had demonstrated the use of ranked data for avoiding the assumptions of normality. The analysis of ranked observations in balanced incomplete block designs having two or three plots per block (paired and triad comparisons) had been given by Kendall and Babigton Smith (1940); Bradley and Terry (1952); Pendergrass and Bradley (1960); Rai (1971) Sadasivan and Rai (1973); Win and Rai (1979) and Gupta and Rai (1980). Rai and Rao (1980) had shown the use of ranked data for solving the problems arising due to heterogeneity of error variances in groups of experiments.

The purpose of the present paper is to outline the procedures where the ranked data can be analysed in place of ordinary analysis of variance when there are two or more criterion of classification. A mathematical model for analysing the ranked data has been developed and the properties and suitability of the model has been studied. This procedure has got two major advantages. Firstly it is quite simple and secondly it is applicable to wider class of data as compared to ordinary analysis of variance.

2. Mathematical Model

The procedure involves first ranking the data in each row of two way table and then testing to see whether the different columns of the resultant of ranks can be supposed to have come from the same universe. If t treatments are compared together in a block, the individual scores are ranked by giving rank 1 to the highest score, 2 to next lower and so on. The smallest score is allotted rank t. Ranking is done afresh for each block and it will have variate values 1, 2,...,t. It is also presumed that treatment T_i has a true rating π_i on a particular subjective continuum throughout the experiment such that

$$\sum_{i=1}^{t} \pi_i = 1$$

On the hypothesis that there is no significant difference among the treatments the difference in the values in each row will arise solely from sampling fluctuations. The rank entered for a particular treatment would then be a matter of chance. In repeated sample, each of the numbers from 1 to t would appear with equal frequency. If the treatments differ significantly, from one another, it would be reflected in the values of π_i. In case of two treatments T_i and T_j the probability that T_i is superior to T_j is given by

$$P(T_i > T_j) = \frac{\pi_i}{\pi_i + \pi_j} \qquad \ldots(1)$$

where $P(T_i > T_j)$ stands for the probability that T_i is prefered to T_j. In case of t treatments $T_1, T_2, \ldots, T_t$ being compared together in a block, the corresponding probability is given by

$$\pi_{12,\ldots,t} = P(T_1 > T_2 > \ldots > T_t)$$

$$= \frac{\pi^{t-}\ \pi_2^{t-2} \ldots \pi_{t-1}}{\Delta_t} \qquad \ldots(2)$$

where

$$\Delta_t = \pi_1^{t-1} (\pi_2^{t-2}\ \pi_3^{t-3} \ldots \pi_{t-1} + \ldots + \pi_t^{t-2} \ldots \pi_3) + \ldots$$

$$+ \pi_t^{t-1} (\pi_1^{t-2}\ \pi_2^{t-3} \ldots \pi_{t-2} + \ldots + \pi_{t-1}^{t-2} \ldots \pi_2) \qquad \ldots(3)$$

It may be seen that sum of the probabilities of all possible combinations of $\pi_{12 \ldots t}$ is unity which indicates that the model indicated at (2) is consistent.

3. The Likelihood Function

We may obtain the likelihood function assuming the probability independence for different replications. The ranks of $T_1, T_2 \ldots, T_t$ in the *k-th* replication will be denoted by $r_{1k}, r_{2k}, \ldots r_{tk}$, respectively. The probability of the specified ranking in the *k-th* replications is given by

$$\pi_1^{t-r_{1k}} \pi_2^{t-r_{2k}} \ldots \pi_t^{t-r_{tk}} \qquad \ldots(4)$$

Multiplying the appropriate expression for all the n replications, we obtain the likelihood function in the general form

$$L = \frac{\overset{t}{\underset{i=1}{\pi}} \pi_t \left(nt \sum_{k=1}^{n} r_{tk} \right)}{(\triangle_t)^n} \qquad \ldots(5)$$

4. Likelihood Ratio Tests and Estimation of Parameters

We can test the significance of the equality of treatment effects. Consider

$H_0 : \pi_1 = \pi_2 = .. \ \pi_t = 1/t$ against the alternative

$H_a : \pi_1 \neq \pi_j$ for some i and j. $i \neq j = 1, 2, \ldots, t$.

The maximum likelihood estimator $p_1 \ldots, p_t$ are obtained by maximising $\log L$ with respect to $\pi_1, \ldots, \pi_t$ subject to the condition that $\sum_{i=1}^{t} \pi_i = 1$. The resulting normal equations are

$$\frac{nt - \Sigma r_{1k}}{p_1} = \frac{n}{D_t} \left[\left\{ (t-1)\, p_1^{t-2} \left(p_2^{t-2} p_3^{t-3} \ldots p_{t-1} + \ldots + p_t^{t-2} p_{t-1}^{t-2} \ldots p_3 \right\} \right) \right.$$

$$\left. + \ldots + p_t^{t-1} \left\{ (t-2) p_1^{t-3} p_3^{t-4} \ldots p_{t-2} + \ldots + p_{t-1}^{t-2} \ldots p_2^2 \right\} \right]$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\frac{nt - \Sigma r_{tk}}{p_t} = \frac{n}{D_t} \left[\left\{ (t-1) p_t^{t-2} p_1^{t-2} (p_1^{t-2} \ldots p_{t-2} + \ldots + p_{t-1}^{t-2} \ldots p_1) \right\} \right.$$

$$\left. + \ldots + p_1^{t-1} \left\{ (t-2) p_t^{t-3} \ldots p_3 + \ldots + p_{t-2}^{t-2} \ldots p_1^2 \right\} \right] \qquad \ldots(6)$$

where D_t is obtained by substituting the values of p_t for π_t in $\triangle_t$. The solution of these equations will give the values of $p_1, \ldots p_t$. The normal equations given in (6) can be solved by iterative methods by taking the initial trial values of p_1 , p_t. If we take the initial trial values of $p_1, p_2, \ldots, p_t$ in proportion to $(\Sigma r_{1k})^{-1} : (\Sigma r_{2k})^{-1} : \ldots : (\Sigma r_{tk})^{-1}$ then the iterative procedure converges quite rapidely.

The likelihood function L given in (5) can be used to obtain the value of likelihood ratio λ and Z which is given by -2 long λ.

$$Z=[2n\,(t-1)-6]\log t-2n\log\,(t-1)!+2\sum_{i=1}^{t}\,(nt-\sum_{k=1} r_{ik})\log p_i -2n\log D_t \qquad \ldots(7)$$

For large n, Z may be taken to have a χ^2—distribution wlth $(t-1)$ degrees of freedom, under the null hypothesis H_0, Wilks (1946). For small sample size, tables for the distribution of Z for various values of t may be developed.

5. TABLE FOR Z

It is possible to generate all combinations of treatment sums of ranks for various treatments and replications. The probability of each such combination may be obtained under the null hypothesis of equality of true treatment ratings. If three items A,B,C are compared in a block, the possible sets of rank sum are 1, 2. 3 ; 1, 3, 2 ; 2, 1, 3 ; 2, 3, 1 ; 3, 1, 2 and 3, 2, 1 with equal probability 1/6. The treatment sum of ranks for two replications and three treatments alongwith the corresponding probabilities are obtained systematically as shown in table 1.

Table 1

The generation of the treatment sums of ranks and proba bilities for three treatments and two replications

Probabilities	*Rank Sums*	*1/6*	*1/6*	*1/6*	*1/6*	*1/6*	*1/6*
		1, 2, 3	*1, 3, 2*	*2, 1, 3*	*2, 3, 1*	*3, 1, 2*	*3, 2, 1*
1/6	1, 2, 3	2, 4, 6	2, 5, 5	3, 3, 6	3, 5, 4	4, 3, 5	4, 4, 4
1/6	1, 3, 2	2, 5, 5	2, 6, 4	3, 4, 5	3, 6, 3	4, 4, 4	4, 5, 3
1/6	2, 1, 3	3, 3, 6	3, 4, 5	4, 2, 6	4. 4, 4	5, 2, 5	5, 3, 4
1/6	2, 3, 1	3, 5, 4	3, 6, 3	4, 4, 4	4, 6, 2	5, 4, 3	5, 5, 2
1/6	3, 1, 2	4, 3, 5	4, 4, 4	5, 2, 5	5, 4, 3	6, 2, 4	6, 3, 3
1/6	3, 2, 1	4, 4, 4	4, 5, 3	5, 3, 4	5, 5, 2	6, 3, 3	6, 4, 2

The combination 2, 5, 5 (say) appears in two places in this table. In row 1 column 2 for example, 2, 5, 5 appears and its probability is 1/36 obtained by multiplying marginal probabilities of corresponding row and column. The probability of the combination is the sum of the individual probabilities and has the value 2/36. When three replications are considered, the generating rows at the top of the table is unchanged, but the columns are replaced by the possible combinations of sums of ranks obtained for two replications with their corresponding probabilities. The procedure is continued for large number of treatments and replications.

The values of sets of treatment sums of ranks are substituted in the normal equations (6) and the estimates of $p_1, p_2, \ldots, p_t$ of $\pi_1, \pi_2, \ldots, \pi_t$ are obtained. The values of Z for different sums of ranks can be obtained from equation (7) and the probabilities can be obtained by following the procedures described above.

6. Combination of Results

Gupta and Rai (1980) have described the procedures of combination of results in case of paired comparison designs. There are different methods for testing the significance depending upon the specification of the alternative hypothesis. Here we will give a procedure which will test the significance of equality of treatment effects and also group $\times$ treatment interaction in case of block designs.

Some times experiments may be conducted at different places or at different time under various circumstances. The experiment may be considered as one of g groups, the u-th of which has n_u replications. Then $n = \sum_{u=1}^{g} n_u$. The failure of treatment parameters $\pi_{1u}, \ldots, \pi_{tu}$ to be the same for each group, represents a group $\times$ treatment interaction or lack of agreement. We now propose a test to detect such interactions.

Consider

$$H_0 : \pi_{iu} = 1/t \text{ for all } i \text{ and } u$$

and

$$H_a : \pi_{iu} \neq 1/t \text{ for some } i \text{ and } u.$$

If λ_c is the likelihood ratio, than we obtain

$$Z_c = -2 \log \lambda_c = \sum_{u=1}^{g} Z_u \qquad \ldots(8)$$

where Z_c is the value of Z given by (7) computed for the u-th group. For large value of n_u, Z_c has the χ^2-distribution with $g(t-1)$ degrees of freedom. This test is used to test the equality of treatment effects. The test of interaction is a test of null hypothesis

$$H_0 : \pi_{iu} = \pi_i \text{ for } i = 1, \ldots, t\ ;\ u = 1, \ldots, g$$

against the alternative

$$H_a : \pi_{t_u} \neq \pi_t \text{ for some } i \text{ and } u.$$

The likelihood ratio test depends upon the value of $Z_c - Z$ and has the χ^2-distribution with $(g-1)$ $(t-1)$ degrees of freedom for large n_u.

7. Appropriateness of the Model

The model is formed by postulating the existence of $t\,!$ probabilities of the type $\pi_{12\ldots t}$ as indicated in section 2. The sum of the probabilities is unity and their maximum likelihood estimators are $f_{12\ldots t}/n$; ..., $f_{t\ldots 21}/n$ for n replications where $f_{12\ldots t}$ is the number of times ranking 1, 2,..., t for treatment $T_1, T_2, \ldots, T_t$ respectively occurs in n replications.

The basic model implies that

H_0 : $\pi_{12\ldots t} = \pi_1^{t-1} \pi_2^{t-2} \ldots \pi_{t-1} / \triangle_t$ for all $\underline{|t}$ arrangements of $\pi_{12\ldots t}$

against the alternative

H_a : $\pi_{12\ldots t} \neq \pi_1^{t-1} \pi_2^{t-2} \ldots \pi_{t-1} / \triangle_t$ for some of the permutations. The general likelihood function is given by

$$L(\pi_{12\ldots t}) = (\pi_{12\ldots t}^{f_{12\ldots t}}) \ldots (\pi_{t\ldots 21}^{f_{t\ldots 21}}) \,. \qquad \ldots(9)$$

If we define $f'_{12\ldots t}$ as the expected frequency corresponding to the observed frequency $f_{12\ldots t}$ then the estimate of the expected frequency under H_0 is given by

$$f'_{12\ldots t} = np_1{}^{t-1} \ldots p_{t-1}/D_t \ldots \qquad \ldots(10)$$

The likelihood ratio statistic λ for testing H_0 is given in terms of frequencies and the value of $-2 \log \lambda$ is given by

$$-2 \log \lambda = 2 \sum^{\underline{|t}} f_{12\ldots t} \log [f_{12\ldots t}/f'_{12\ldots t}] \qquad \ldots(11)$$

For large n, this statistic has a x^2 distribution with $(\underline{|t} - t)$ degrees of freedom. By applying transformation in expression (11) as suggested by Rai (1971), it can be shown that

$$-2 \log \lambda \approx \Sigma (f_{12\ldots t} - f'_{12\ldots t})^2 / f'_{12\ldots t} \qquad \ldots(12)$$

Thus the statistic 2 log λ can be transformed to the usual χ^2 test of goodness of fit.

8. An Illustrative Example

Some of the procedures developed in this paper will be demonstrated by the numerical example given below :

TABLE 2

Frequencies of rankings with t=4 and n=50

$f_{1234}=3$	(2.8)	$f_{2341}=1$	(1.8)	$f_{3412}=2$	(2.2)
$f_{1243}=2$	(2.2)	$f_{2314}=2$	(1.6)	$f_{3421}=2$	(1.6)
$f_{1324}=8$	(6.4)	$f_{2413}=2$	(2.8)	$f_{4123}=1$	(1.4)
$f_{1342}=6$	(5.6)	$f_{2431}=2$	(1.8)	$f_{4132}=1$	(1.2)
$f_{1423}=1$	(0.8)	$f_{3124}=2$	(2.4)	$f_{4213}=1$	(1.8)
$f_{1432}=1$	(0.6)	$f_{3142}=2$	(2.0)	$f_{4231}=2$	(1.2)
$f_{2134}=4$	(4.4)	$f_{3214}=1$	(1.2)	$f_{4312}=1$	(1.2)
$f_{2143}=1$	(1.4)	$f_{3241}=1$	(0.8)	$f_{4321}=1$	(0.8)

From the above table we obtain the following preference matrix:

TABLE 3

Preference Matrix and sum of ranks.

Treatment No.	*Number of times ranked as*				*Sum of ranks* Σr_i
	First	*Second*	*Third*	*Fourth*	
1	21	12	10	7	103
2	11	10	19	10	128
3	8	15	13	14	133
4	10	13	8	19	136

From the above table, we can obtain the values of p_1, p_2, p_3 and p_4 alongwith the value of Z. These values are given below :

$$p_1=0.3032$$
$$p_2=0.2486$$
$$p_3=0.2328$$
$$p_4=0.2154$$
$$Z=20.71$$

The value of Z taken as χ^2 with 3 degrees of freedom indicates highly significant treatment.

In order to test the appropriateness of the model, we may apply the goodness of fit test. The different value of expected frequencies are obtained from (10) and have been shown in paranthesis in Table 2. Using (12) we find the value of $-2 \log \lambda = 6.23$ which is distributed like χ^2 with 20 d.f. This value of χ^2 is not significant which indicates that the proposed model is quite satisfactory for these data.

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